

Constructive ϕ^4 field theory without tears

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Abstract

We propose to treat the ϕ^4 Euclidean theory constructively in a simpler way. Our method, based on a new kind of "loop vertex expansion", no longer requires the painful intermediate tool of cluster and Mayer expansions.

1 Introduction

Constructive field theory build functions whose Taylor expansion is perturbative field theory [1, 2]. Any formal power series being asymptotic to infinitely many smooth functions, perturbative field theory alone does not give any well defined mathematical recipe to compute to arbitrary accuracy any physical number, so in a deep sense it is no theory at all.

In field theory "thermodynamic" or infinite volume quantities are expressed by connected functions. One main advantage of perturbative field theory is that connected functions are simply the sum of the connected Feynman graphs. But the expansion diverges because there are too many such graphs. However to know connectedness does not require the full knowledge of a Feynman graph (with all its loop structure) but only the (classical) notion of a spanning tree in it. This remark is at the core of the developments of constructive field theory, such as cluster expansions, summarized in the constructive golden rule:

"Thou shall not know most of the loops, or thou shall diverge!"

Some time ago Fermionic constructive theory was quite radically simplified. It was realized that it is possible to rearrange perturbation theory *order by order* by grouping together pieces of Feynman graphs which share a common tree [3, 4]. This is made easily with the help of a universal combinatoric so-called forest formula [5, 6] which once and for all essentially solves the problem that a graph can have many spanning trees. Indeed it splits any amplitude of any connected graph in a certain number of pieces and attributes them in a "democratic" and "positivity preserving" way between all its spanning trees. Of course

the possibility for such a rearrangement to lead to convergent resummation of Fermionic perturbation theory ultimately stems from the Pauli principle which is responsible for *analyticity* of that expansion in the coupling constant.

Using this formalism Fermionic theory can now be manipulated at the constructive level almost as easily as at the "perturbative level to all orders". It lead to powerful mathematical physics theorems such as for instance those about the behavior of interacting Fermions in 2 dimensions [7, 8, 9], and to more explicit constructions [10] of just renormalizable Fermionic field theories such as the Gross-Neveu model in two dimensions first built in [11, 12].

But bosonic constructive theory remained awfully difficult. To compute the thermodynamic functions, until today one needed to introduce two different expansions one of top of the other. The first one, based on a discretization of space into a lattice of cubes which breaks the natural rotation invariance of the theory, is called a cluster expansion. The result is a dilute lattice gas of clusters but with a remaining hardcore interaction. Then a second expansion called Mayer expansion removes the hardcore interaction. The same tree formula is used *twice* once for the cluster and once for the Mayer expansion¹, the breaking of rotation invariance to compute rotation invariant quantities seems *ad hoc* and the generalization of this technique to many renormalization group steps is considered so difficult that despite courageous attempts towards a better, more explicit formalization [14, 15], it remains until now confined to a small circle of experts.

The bosonic constructive theory cannot be simply rearranged in a convergent series *order by order* as in the Fermionic case, because all graphs at a given order have the same sign. Perturbation theory has zero convergence radius for bosons. The oscillation which allows resummation (but only e.g. in the Borel sense) of the perturbation theory must take place between infinite families of graphs of different orders. To explicitly identify such families and rearrange the perturbation theory accordingly seemed until now very difficult. The cluster and Mayer expansion perform this task but in a very complicated and indirect way.

In this paper we at last identify such infinite families of graphs. They give rise to an explicit convergent expansion for the connected functions of bosonic ϕ^4 theory, without any lattice and cluster or Mayer expansion. In fact we stumbled upon this new method by trying to adapt former cluster expansions to large matrix ϕ^4 models in order to extend constructive methods to non-commutative field theory (see [16] for a recent review). The matrix version is described in a separate publication [17]. Hopefully it should allow a non-perturbative construction of the ϕ^{*4} theory on Moyal space \mathbb{R}^4 , whose renormalizable version was pioneered by Grosse and Wulkenhaar [18].

¹It is possible to combine both expansions into a single one [13], but the result cannot be considered a true simplification.

2 The example of the pressure of ϕ^4

We take as first example the construction of the pressure of ϕ_4^4 in a renormalization group (RG) slice. The goal is e.g. to prove its Borel summability in the coupling constant uniformly in the slice index, without using any lattice (breaking Euclidean invariance) nor any cluster or Mayer expansion.

The propagator in a RG slice j is e.g.

$$C_j(x, y) = \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2} e^{-(x-y)^2/4\alpha} \alpha^{-2} d\alpha \leq K M^{2j} e^{-cM^j|x-y|} \quad (1)$$

where M is a constant defining the size of the RG slices, and K and c from now on are generic names for inessential constants, respectively large and small. We could also use compact support cutoffs in momentum space to define the RG slices.

Consider a local interaction $\lambda \int \phi^4(x) d^4x = \lambda \text{Tr} \phi^4$ where the trace means spatial integration. For the moment assume the coupling λ to be real positive and small. We decompose the ϕ^4 functional integral according to an intermediate field as:

$$\int d\mu_{C_j}(\phi) e^{-\lambda \text{Tr} \phi^4} = \int d\nu(\sigma) e^{-\frac{1}{2} \text{Tr} \log(1+iH)} \quad (2)$$

where $d\nu$ is the ultralocal measure on σ with covariance $\delta(x-y)$, and $H = \lambda^{1/2} D_j \sigma D_j$ is an Hermitian operator, with $D_j = C_j^{1/2}$.

The pressure is known to be the Borel sum of all the connected vacuum graphs with a particular root vertex fixed at the origin. We want to prove this through a new method.

We define the *loop vertex*² $V = -\frac{1}{2} \text{Tr} \log(1+iH)$. This loop vertex can be pictured as in the left hand side of Figure 1. The trace means integration over a “root” x_0 . Cyclic invariance means that this root can be moved everywhere over the loop. It is also convenient to also introduce an arrow, by convention always turning counterclockwise for a $+iH$ convention, and anti-clockwise for a complex conjugate loop vertex $\bar{V} = -\frac{1}{2} \text{Tr} \log(1-iH)$.

We then expand the exponential as $\sum_n \frac{V^n}{n!}$. To compute the connected graphs we give a (fictitious) index v , $v = 1, \dots, n$ to all the σ fields of a given loop vertex V_v . This means that we consider n different copies σ_v of σ with a degenerate Gaussian measure $d\nu(\{\sigma_v\})$ whose covariance is $\langle \sigma_v \sigma_{v'} \rangle_\nu = \delta(x-y)$. The functional integral over $d\nu(\sigma)$ is equal to the functional integral over $d\nu(\{\sigma_v\})$. We apply then the forest formula of [6] to test connexions between the loop vertices from 1 to n . (The lines of this forest, which join loop vertices correspond to former ϕ^4 vertices.)

The logarithm of the partition function $\log Z(\Lambda)$ at finite volume Λ is given by this formula restricted to trees (like in the Fermionic case [4]), and spatial integration restricted to Λ . The pressure or infinite volume limit of $\frac{\log Z(\Lambda)}{|\Lambda|}$ is given by the same *rooted* tree formula but with one particular position fixed at

²To avoid any confusion with the former ϕ^4 vertices we shall not omit the word *loop*.

the origin, for instance the position associated to a particular root line ℓ_0 . More precisely:

Theorem 2.1.

$$\lim_{\Lambda \rightarrow \mathbb{R}^4} \frac{\log Z(\Lambda)}{|\Lambda|} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_T \left\{ \prod_{\ell \in T} \left[\int_0^1 dw_\ell \right] \right\} G_T(\sigma, x_{\ell_0})|_{x_{\ell_0}=0} \quad (3)$$

$$G_T(\sigma, x_{\ell_0}) = \prod_{\ell \in T} \int d^4 x_\ell d^4 y_\ell \int d\nu_T(\{\sigma_v\}, \{w\}) \left\{ \prod_{\ell \in T} \left[\delta(x_\ell - y_\ell) \frac{\delta}{\delta \sigma_{v(\ell)}(x_\ell)} \frac{\delta}{\delta \sigma_{v'(\ell)}(y_\ell)} \right] \right\} \prod_v V_v, \quad (4)$$

where

- each line ℓ of the tree joins two different vertices $V_{v(\ell)}$ and $V_{v'(\ell)}$ at point x_ℓ and y_ℓ , which are identified through the function $\delta(x_\ell - y_\ell)$ (since the covariance of σ is ultralocal),
- the sum is over rooted trees over n vertices, which have therefore $n - 1$ lines, with root ℓ_0 ,
- the normalized Gaussian measure $d\nu_T(\{\sigma_v\}, \{w\})$ over the vector field σ_v has covariance

$$\langle \sigma_v, \sigma_{v'} \rangle = \delta(x - y) w^T(v, v', \{w\})$$

where $w^T(v, v', \{w\})$ is 1 if $v = v'$, and the infimum of the w_ℓ for ℓ running over the unique path from v to v' in T if $v \neq v'$. This measure is well-defined because the matrix w^T is positive.

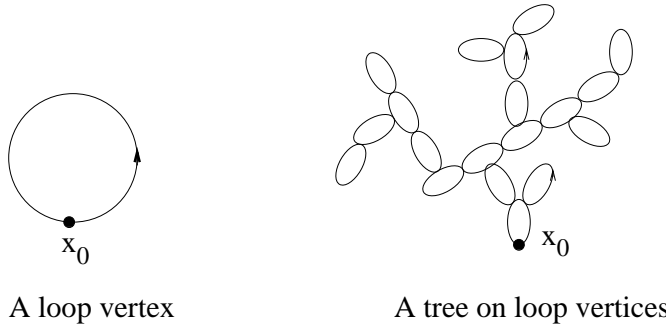


Figure 1: Loop vertices and a tree on them

This is indeed the outcome of the universal tree formula of [6] in this case. To check it, we need only to move by cyclicity the local root of each loop nearest to the global root in the tree. This global root point is chosen for simplicity in

formulas above at a particular root line ℓ_0 , but in fact it could be fixed anywhere in an arbitrarily chosen “root loop”, as shown on the right hand side of Figure 1 (with all loops oriented counterclockwise).

But there is an other representation of the same object. A tree on connecting loops such as the one shown in the right hand side of Figure 1 can also be drawn as a set of dotted lines dividing in a *planar* way a *single loop* as in Figure 2. Each dotted line carries a $\delta(x_\ell - y_\ell)$ function which identifies pairs of points on the border of the loop joined by the dotted line, and is equipped with a coupling constant, because it corresponds to an old ϕ^4 vertex. This second picture is obtained by turning around the tree. The pressure corresponds to the sum over such planar partitions of a single big loop with an arbitrary root point fixed at the origin. The corresponding interpolated measure $d\nu$ can be described also very simply in this picture. There is now a σ_v field copy for every domain v inside the big loop, a w parameter for each dotted line, and the covariance of two σ_v and $\sigma_{v'}$ fields is the ordinary δ function covariance multiplied by a weakening parameter which is the infimum of the w parameters of the dotted lines one has to *cross* to go from v to v' . The counterclockwise orientation of the big loop corresponds to the $+iH$ convention.

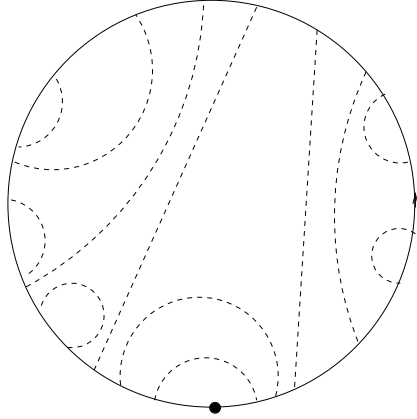


Figure 2: The big loop representation

In this new picture we see indeed many loops... but the golden rule is not violated. In this new representation it simply translates into

“Thou shall see only planar (or genus-bounded) structures...”

(Recall that genus-bounded graphs are not many and don’t make perturbation theory diverge.)

Let us prove now that the right hand side of formula (3) is convergent as series in n .

Theorem 2.2. *The series (3) is absolutely convergent for λ small enough, and the sum is bounded by KM^{4j} .*

Proof We shall use the first representation of Figure 1. Consider a loop vertex V_v of coordination k_v in the tree. Let us compute more explicitly the outcome of the k_v derivatives $\prod_{i=1}^{k_v} \frac{\delta}{\delta \sigma(x_i)}$ acting on

$$V = -\frac{1}{2} \text{Tr} \log(1 + iH)$$

which created this loop vertex.

Consider the operator

$$C_j(\sigma) = D_j \frac{1}{1 + iH} D_j. \quad (5)$$

Calling x_1 the root position for the loop vertex V_v , that is the unique position from which a path goes to the root of T , the loop vertex factor V_v after action of the derivatives is

$$\left[\prod_{i=1}^{k_v} \frac{\delta}{\delta \sigma(x_i)} \right] V_v = \frac{1}{2} (-i\sqrt{\lambda})^{k_v} \sum_{\tau} \prod_{i=1}^{k_v} C_j(\sigma, x_{\tau(i)}, x_{\tau(i+1)}) \quad (6)$$

where the sum is over all permutations τ of $[2, \dots, k]$, completed by $\tau(1) = \tau(k+1) = 1$.

To bound the integrals over all positions except the root, we need only a very simple lemma:

Lemma 2.1. *There exists K such that for any x and any v*

$$|[C_j(\sigma_v)]^{k_v}(x, x)| \leq K^{k_v} M^{(4-2k_v)j} \quad \forall \sigma_v. \quad (7)$$

Since iH is anti-hermitian we have $\|(1 + iH)^{-1}\| \leq 1$. It is obvious from (1) that $\|C_j\| \leq KM^{-2j}$, hence $\|D_j\| \leq KM^{-j}$. We have

$$[C_j(\sigma_v)]^{k_v}(x, x) = \int dy dz D_j(x, y) A(y, z) D_j(z, x) = \langle f, Af \rangle \quad (8)$$

for $f = D_j(x, \cdot)$ and $A = (1 + iH)^{-1} [C_j(1 + iH)^{-1}]^{k_v-1}$. The norm of the operator A is bounded by $K^{k_v-1} M^{-2j(k_v-1)}$. Since $\|f\|^2 \leq KM^{2j}$, the result follows. \square

To bound the dx_ℓ integrals we start from the leaves and insert the bound (7), which also means that the multiplication operator $[C_j(\sigma_v)]^{k_v}(x, x)$ (diagonal in x space) has a norm bounded by $K^{k_v} M^{(4-2k_v)j}$ uniformly in σ . We then progress towards the root. By induction, multiplying norms, adding the $\frac{1}{2}(-i\sqrt{\lambda})^{k_v}$ factors from (6) and taking into account the factorials from the sum over the permutations τ in (6) gives exactly

$$\prod_v \frac{1}{2} (k_v - 1)! \lambda^{k_v/2} K^{k_v} M^{4j-2jk_v}. \quad (9)$$

For a tree on n loop vertices $\sum_v k_v = 2(n-1)$ hence $\sum_v (4-2k_v) = 4n - 4(n-1) = 4$ so that collecting all dimensional factors we get a M^{4j} global n

independent factor as should be the case for vacuum graphs in the ϕ^4 theory in a single RG slice.

We can now integrate the previous bound over the complicated measure $d\nu_T$ and over the $\{w_\ell\}$ parameters. But since our bound is independent of σ^v , since the measure $d\nu(\sigma)$ is normalized, and since each w_ℓ runs from 0 to 1, this does not change the result.

Finally by Cayley's theorem the sum over trees costs $\frac{n!}{\prod_v (k_v - 1)!}$. The $n!$ cancels with the $1/n!$ of (3) and the $1/(k_v - 1)!$ exactly cancel the one in (9). It remains a geometric series bounded by $\frac{1}{2}M^{4j}(\lambda K)^{n-1}$ hence convergent for small λ , and the sum is bounded by $K.M^{4j}$. \square

3 Uniform Borel summability

Rotating to complex λ and Taylor expanding out a fixed number of ϕ^4 vertices proves Borel summability in λ *uniformly in j* .

Definition *A family f_j of functions is called **Borel summable** in λ uniformly in j if*

- *Each f_j is analytic in a disk $D_R = \{\lambda | \operatorname{Re} \lambda^{-1} > 1/R\}$;*
- *Each f_j admits an asymptotic power series $\sum_k a_{j,k} \lambda^k$ (its Taylor series at the origin) hence:*

$$f_j(\lambda) = \sum_{k=0}^{r-1} a_{j,k} \lambda^k + R_{j,r}(\lambda) \quad (10)$$

such that the bound

$$|R_{j,r}(\lambda)| \leq A_j \rho^r r! |\lambda|^r \quad (11)$$

holds uniformly in r and $\lambda \in D_R$, for some constant $\rho \geq 0$ independent of j and constants $A_j \geq 0$ which may depend on j .

Then every f_j is Borel summable [19], i.e. the power series $\sum_k a_{j,k} \frac{t^k}{k!}$ converges for $|t| < \frac{1}{\rho}$, it defines a function $B_j(t)$ which has an analytic continuation in the j independent strip $S_\rho = \{t | \operatorname{dist}(t, \mathbb{R}^+) < \frac{1}{\rho}\}$. Each such function satisfies the bound

$$|B_j(t)| \leq B_j e^{\frac{t}{\rho}} \quad \text{for } t \in \mathbb{R}^+ \quad (12)$$

for some constants $B_j \geq 0$ which may depend on j . Finally each f_j is represented by the following absolutely convergent integral:

$$f_j(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{t}{\lambda}} B_j(t) dt \quad \text{for } \lambda \in C_R. \quad (13)$$

Theorem 3.1. *The series for the pressure is uniformly Borel summable with respect to the slice index.*

Proof It is easy to obtain uniform analyticity for $\text{Re } \lambda > 0$ and $|\lambda|$ small enough, a region which obviously contains a disk D_R . Indeed all one has to do is to reproduce the previous argument but adding that for H Hermitian, the operator $(1 + ie^{i\theta}H)^{-1}$ is bounded by $\sqrt{2}$ for $|\theta| \leq \pi/4$. Indeed if $\pi/4 \leq \text{Arg } z \leq 3\pi/4$, we have $|(1 + iz)^{-1}| \leq \sqrt{2}$.

Then the uniform bounds (11) follow from expanding the product of resolvents in (6) up to order $r - 2(n - 1)$ in λ by an explicit Taylor formula with integral remainder followed by explicit Wick contractions. The sum over the contractions leads to the $\rho^r r!$ factor in (11). \square

4 Connected functions and their decay

To obtain the connected functions with external legs we need to add resolvents to the initial loop vertices. A resolvent is an operator $C_j(\sigma_r, x, y)$. The connected functions $S^c(x_1, \dots, x_{2p})$ are obtained from the normalized functions by the standard procedure. We have the analog of formula (3) for these connected functions:

Theorem 4.1.

$$\begin{aligned}
S^c(x_1, \dots, x_{2p}) &= \sum_{\pi} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_T \left\{ \prod_{\ell \in T} \left[\int_0^1 dw_{\ell} \int d^4 x_{\ell} d^4 y_{\ell} \right] \right\} \\
&\int d\nu_T(\{\sigma_v\}, \{\sigma_r\}, \{w\}) \left\{ \prod_{\ell \in T} \left[\delta(x_{\ell} - y_{\ell}) \frac{\delta}{\delta\sigma_{v(\ell)}(x_{\ell})} \frac{\delta}{\delta\sigma_{v'(\ell)}(y_{\ell})} \right] \right\} \\
&\prod_v V_v \prod_{r=1}^p C_j(\sigma_r, x_{\pi(r,1)}, x_{\pi(r,2)}), \tag{14}
\end{aligned}$$

where

- the sum over π runs over the pairings of the $2p$ external variables into pairs $(x_{\pi(r,1)}, x_{\pi(r,2)})$, $r = 1, \dots, p$,
- each line ℓ of the tree joins two different loop vertices or resolvents $V_{v(\ell)}$ and $V_{v'(\ell)}$ at point x_{ℓ} and y_{ℓ} , which are identified through the function $\delta(x_{\ell} - y_{\ell})$ because the covariance of σ is ultralocal,
- the sum is over trees joining the $n + p$ loop vertices and resolvents, which have therefore $n + p - 1$ lines,
- the measure $d\nu_T(\{\sigma_v\}, \{\sigma_r\}, \{w\})$ over the $\{\sigma\}$ fields has covariance $\langle \sigma_{\alpha}, \sigma_{\alpha'} \rangle = \delta(x - y) w^T(\alpha, \alpha', \{w\})$ where $w^T(\alpha, \alpha', \{w\})$ is 1 if $\alpha = \alpha'$ (where $\alpha, \alpha' \in \{v\}, \{r\}$), and the infimum of the w_{ℓ} for ℓ running over the unique path from α to α' in T if $\alpha \neq \alpha'$. This measure is well-defined because the matrix w^T is positive.

Now we want to prove not only convergence of this expansion but also scaled tree decay between external arguments:

Theorem 4.2. *The series (14) is absolutely convergent for λ small enough, its sum is uniformly Borel summable in λ and we have:*

$$|S^c(z_1, \dots, z_{2p})| \leq (2p)! K^p |\lambda|^{p-1} M^{2pj} e^{-cM^j d(z_1, \dots, z_{2p})} \quad (15)$$

where $d(z_1, \dots, z_{2p})$ is the length of the shortest tree which connects all the points z_1, \dots, z_p .

The proof of convergence (and of uniform Borel summability) is similar to the one for the pressure.

The tree decay (15) is well known and standard to establish through the traditional cluster and Mayer expansion. It is due to the existence of a tree of C_j propagators between external points in any connected function. In the present expansion, this tree is hidden in the resolvents and loop vertices, so that an expansion on these resolvents (and loop vertices) is necessary in one form or another to prove (15). It does not seem to follow from bounds on operator norms only: the integral over the σ field has to be bounded more carefully.

The standard procedure to keep resolvent expansions convergent is a so-called large/small field expansion on σ . In the region where σ is small the resolvent expansion converges. In the large field region there are small probabilistic factors coming from the $d\nu_T$ measure. This is further sketched in subsection 5.2.

However the large/small field expansion again requires a discretization of space into a lattice: a battery of large/small field tests is performed, on the average of the field σ over each cube of the lattice. We prefer to provide a new and different proof of (15). It relies on a single resolvent step followed by integration by parts, to establish a Fredholm inequality on the modulus square of the $2p$ point function. From this Fredholm inequality the desired decay follows easily. The rest of this section is devoted to the proof of (15) in the simplest case $p = 1$. The most general case is sketched in subsection 5.1.

The two point function S^c is simply called $S(x, y)$ from now on, and for $p = 1$ (15) reduces to

$$|S(x, y)| \leq K M^{2j} e^{-cM^j |x-y|}. \quad (16)$$

We work with n , T and $\{w\}$ fixed in (14). We use the resolvent as root for T , from which grow q subtrees T_1, \dots, T_q . In more pictorial terms, (14) represents a chain of resolvents from x to y separated by insertions of q subtrees. Figure 3 is therefore the analog of Figure 1 in this context³.

A representation similar to the big loop of Figure 2 pictures the decorated resolvent as a half-circle going from x to y , together with a set of planar dotted lines for the vertices. The $+i$ convention again corresponds to a particular orientation. For reason which should become clear below, we picture the planar dotted lines all on the same side of the x - y line, hence *inside the half-disk*.

To each such drawing, or graph G , there is an associated Gaussian measure $d\nu_G$ which is the one from which the drawing came as a tree. Hence it has a field

³A similar figure is a starting point for the 1PI expansion of the self-energy in [7, 9].

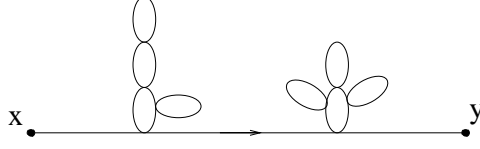


Figure 3: Three resolvents with two branching subtrees

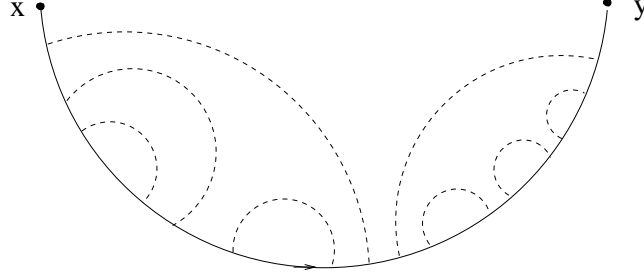


Figure 4: The half-circle representation of Figure 3

copy associated to each planar region of the picture, a weakening parameter w associated to each dotted line, and the covariance between the σ fields of different regions is given by the infimum over the parameters of the dotted lines that one has to cross to join these two regions.

There is also for each such G an *amplitude*. Let us write simply $\int d\nu_G$ for the normalized integral $\int_0^1 \prod_{\ell \in G} dw_\ell \int d\nu_G(\{\sigma\}, \{w\})$. If the graph has n dotted lines hence $2n + 1$ resolvents from x to y , its amplitude is

$$A_G(x, y) = \lambda^n \int d\nu_G \int \left[\prod_{\ell \in G} d^4 x_\ell \right] \prod_{i=1}^{2n+1} C_j(\sigma_i, x_{i-1}, x_i) \quad (17)$$

where the product over ℓ runs over the dotted lines and the product over i runs over the resolvents along the half-circle, with $x_0 = x$ and $x_{2n+1} = y$. σ_i is the field copy of the region just before point x_i and the $2n$ positions x_1, \dots, x_{2n} are equal in pairs to the n corresponding x_ℓ 's according to the pairings of the dotted lines.

We shall prove

Lemma 4.1. *There exists some constant K such that for λ small enough*

$$\sup_{G, n(G)=n} |A_G(x, y)| \leq (|\lambda|K)^{n/2} M^{2j} e^{-cM^j |x-y|}. \quad (18)$$

From this Lemma (16) obviously follows. Indeed the remaining sum over Cayley trees costs at most $K^n n!$, which is compensated by the $\frac{1}{n!}$ in (14). In the language of planar graphs the planar dotted lines cost only K^n . Hence the sum over n converges for λ small enough because of the $|\lambda|^{n/2}$ factor in

(18). Remark that this factor $|\lambda|^{n/2}$ is not optimal; $|\lambda|^n$ is expected; but it is convenient to use half of the coupling constants for auxiliary sums below.

We apply a Schwarz inequality to $|A_G(x, y)|^2$, relatively to the normalized measure $d\nu_G$:

$$|A_G(x, y)|^2 \leq A_{G \cup \bar{G}}(x, y), \quad (19)$$

$$A_{G \cup \bar{G}}(x, y) = \int d\nu_G \int \left[\prod_{\ell \in G} d^4 x_\ell d^4 \bar{x}_\ell \right] \prod_{i=1}^{2n+1} C_j(\sigma_i, x_{i-1}, x_i) \bar{C}_j(\sigma_i, \bar{x}_{i-1}, \bar{x}_i) \quad (20)$$

with hopefully straightforward notations.

The quantity on the right hand side is now pointwise positive for any σ . It can be considered as the amplitude $A_{G \cup \bar{G}}(x, y)$ associated to a *mirror graph* $G \cup \bar{G}$. Such a mirror graph is represented by a full disk, with x and y diametrically opposite, and no dotted line crossing the corresponding diameter. The upper half-circle represents the complex conjugate of the lower part. Hence the upper half-disk is exactly the mirror of the lower half-disk, with orientation reversed, see Figure 5.

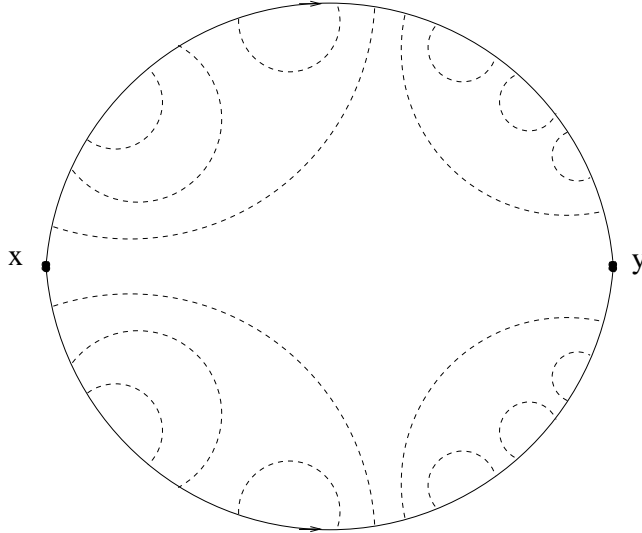


Figure 5: The mirror graph $G \cup \bar{G}$ for the graph G of Figure 4

The Gaussian measure associated to such a mirror graph remains that of G , hence it has a single weakening w parameter for each dotted line and its mirror line, and it has a single copy of a σ field for each *pair* made of a region of the disk and its mirror region. Let's call such a pair a "mirror region". The covariance between two fields belonging to two mirror regions is again the infimum of the

w parameters crossed from one region to the other, but e.g. staying entirely in the lower half-disk (or the upper half-disk).

We shall now perform a single resolvent expansion step and integration by parts, together with a bound which reproduces an amplitude similar to $A_{G \cup \bar{G}}$. The problem is that the category of mirror graphs is not exactly stable in this operation; this bound generates other graphs with “vertical” dotted lines between the lower and upper half of the circle. To prove our bound inductively we need therefore to generalize slightly the class of *mirror graphs* and their associated Gaussian measures to a larger category of graphs $G \cup \bar{G} \cup V$, called *generalized mirror graphs* or GM graphs and pictured in Figure 6. They are identical to mirror graphs except that they can have in addition a certain set V of “vertical” dotted lines between the lower and upper half of the circle, again without any crossing.

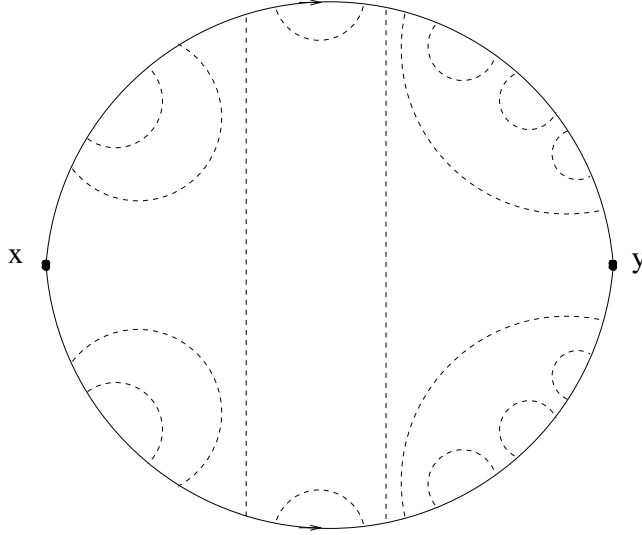


Figure 6: The generalized mirror graphs

There is a corresponding measure $d\nu_{G,V}$ with similar rules; there is a single w parameter for each pair of dotted line and its mirror, in particular there is a w parameter for each vertical line. Again the covariance between two fields belonging to two mirror regions is the infimum of the w parameters crossed from one mirror region to the other, *staying entirely in e.g. the lower half-disk*. The upper half-part is still the complex conjugate of the lower half-part. The order of a GM graph is again the total number $L = 2n + |V|$ of dotted lines and its amplitude is given by a pointwise positive integral similar to (20):

$$\begin{aligned}
A_{G \cup \bar{G} \cup V}(x, y) &= \lambda^L \int d\nu_{G \cup V} \int \left[\prod_{\ell \in G} d^4 x_\ell d^4 \bar{x}_\ell \right] \left[\prod_{\ell \in V} dy_\ell \right] \\
&\quad \prod_{i=1}^{2n+|V|+1} C_j(\sigma_i, z_{i-1}, z_i) \bar{C}_j(\sigma_i, \bar{z}_{i-1}, \bar{z}_i), \quad (21)
\end{aligned}$$

where the z 's and \bar{z} 's are either x_ℓ 's, \bar{x}_ℓ 's or y_ℓ 's according to the graph.

Defining the integrand $I_{G \cup \bar{G} \cup V}(x, y)$ of a GM graph so that $A_{G \cup \bar{G} \cup V}(x, y) = \int d\nu_{G \cup V} I_{G \cup \bar{G} \cup V}(x, y)$, we have:

Lemma 4.2. *For any GM graph we have, uniformly in σ , x and y :*

$$I_{G \cup \bar{G} \cup V}(x, y) \leq (K|\lambda|)^L M^{4j}. \quad (22)$$

Indeed the quantity $I_{G \cup \bar{G} \cup V}(x, y)$ is exactly the same than a pressure graph but with two fixed points and some propagators replaced by complex conjugates, hence the proof through the norm estimates of Lemma 2.1 is almost identical to the one of Theorem 2.2.

We now write the resolvent step which results in an integral Fredholm inequality for the supremum of the amplitudes of any generalized mirror graph.

Let us define the quantity

$$\Gamma_L(x, y) = \sup_{\text{GM graphs } G, V \mid L(G)=L} |\lambda|^{-L/2} A_{G \cup \bar{G} \cup V}(x, y). \quad (23)$$

We shall prove by induction on L :

Lemma 4.3. *There exists some constant K such that for λ small enough*

$$\Gamma_L(x, y) \leq KM^{4j} \left(e^{-cM^j|x-y|} + |\lambda|^{1/2} \int dz e^{-cM^j|x-z|} \Gamma_L(z, y) \right). \quad (24)$$

From that lemma indeed obviously follows

Lemma 4.4. *There exists some constant K such that for λ small enough*

$$\Gamma_L(x, y) \leq KM^{4j} e^{-cM^j|x-y|}. \quad (25)$$

Indeed iterating the integral Fredholm equation (24) leads obviously to (25).

Taking (21) and (23) into account to reinstall the $\lambda^{L/2}$ factor, considering the equation $L = 2n + V$ and taking a square root because of (19), Lemma 4.1 is then nothing but Lemma 4.4 for the particular case $V = 0$.

The rest of this section is therefore devoted to the proof of Lemma 4.3, by a simple induction on L .

If $L = 0$, $\Gamma_0(x, y) = \int d\nu C_j(\sigma, x, y) \bar{C}_j(\sigma, x, y)$. Expanding the $C_j(\sigma, x, y)$ propagator, we get

$$\Gamma_0(x, y) = \int d\nu [C_j(x, y) - i\sqrt{\lambda} \int dz C_j(x, z) \sigma(z) C_j(\sigma, z, y)] \bar{C}_j(\sigma, x, y). \quad (26)$$

For the first term $|\int d\nu C_j(x, y) \bar{C}_j(\sigma, x, y)|$, we simply use bounds (1) and (22) in the case $L = 0$. For the second term we Wick contract the σ field (i.e. integrate by parts over σ). There are two subcases: the Wick contraction $\frac{\delta}{\delta\sigma}$ hits either $C_j(\sigma, z, y)$ or $\bar{C}_j(\sigma, x, y)$. We then apply the inequality

$$|ABC| \leq \frac{A}{2}(M^{2j}|B|^2 + M^{-2j}|C|^2), \quad (27)$$

which is valid for any positive A . In the first subcase we take $A = \int dz C_j(x, z)$, $B = C_j(\sigma, z, y)$ and $C = C_j(\sigma, z, z) \bar{C}_j(\sigma, x, y)$, hence write

$$\begin{aligned} & \left| \int dz C_j(x, z) C_j(\sigma, z, z) C_j(\sigma, z, y) \bar{C}_j(\sigma, x, y) \right| \leq \\ & \int dz \frac{C_j(x, z)}{2} [M^{2j}|C_j(\sigma, z, y)|^2 + M^{-2j}|C_j(\sigma, z, z) \bar{C}_j(\sigma, x, y)|^2] \end{aligned} \quad (28)$$

and in the second subcase we write similarly

$$\begin{aligned} & \left| \int dz C_j(x, z) C_j(\sigma, z, y) \bar{C}_j(\sigma, x, z) \bar{C}_j(\sigma, z, y) \right| \leq \\ & \int dz \frac{C_j(x, z)}{2} [M^{2j}|C_j(\sigma, z, y)|^2 + M^{-2j}|\bar{C}_j(\sigma, x, z) \bar{C}_j(\sigma, z, y)|^2]. \end{aligned} \quad (29)$$

Using the uniform bound (22) on the “trapped loop” $|C_j(\sigma, z, z)|^2$ or $|\bar{C}_j(\sigma, x, z)|^2$ in the C term we obtain

$$\begin{aligned} \Gamma_0(x, y) & \leq KM^{4j}e^{-cM^j|x-y|} + |\lambda|K \left(\Gamma_0(x, y) \right. \\ & \quad \left. + M^{4j} \int dz e^{-cM^j|x-z|} \Gamma_0(z, y) \right) \end{aligned} \quad (30)$$

so that (24) hence Lemmas 4.3 and 4.4 hold for $L = 0$.

We now assume that (24), hence also (25), is true up to order L and we want to prove (24) at order $L + 1$. Consider a GM graph of order $L + 1$. If $V \geq 1$ we can decompose it as a convolution of smaller GM graphs:

$$A_{G \cup \bar{G} \cup V}(x, y) = \lambda \int dy_1 A_{G_1 \cup \bar{G}_1}(x, y_1) A_{G_2 \cup \bar{G}_2 \cup V_2}(y_1, y) \quad (31)$$

with total orders L_1 for G_1 and L_2 for $G_2, V_2 = V - \{1\}$ strictly smaller than $L + 1$. Applying the induction hypothesis (25) to these smaller GM graphs we get directly that

$$\sup_{G, V | L(G \cup \bar{G} \cup V) = L+1, V > 0} |\lambda|^{-(L+1)/2} A_{G \cup \bar{G} \cup V}(x, y) \leq KM^{4j}e^{-cM^j|x-y|}. \quad (32)$$

Hence we have now only to prove (24) for mirror graphs with $V = \emptyset$. Consider now such a mirror graph G . Because of the $|\lambda|^{-L/2}$ in (23), we should remember that we have only a remaining factor $|\lambda|^{L/2}$ to use for our bounds on Γ_L .

Starting at x we simply expand the first resolvent propagator $C_j(\sigma, x, x_1)$ as $C_j(x, x_1) - \int dz C_j(x, z) i\sqrt{\lambda}\sigma(z) C_j(\sigma, z, x_1)$.

For the first term we call x_{i_1} the point to which x_1 is linked by a dotted line and apply a Schwarz inequality of the (27) type, with:

$$\begin{aligned} A &= \int dx_1 C_j(x, x_1), \\ B &= \int \prod_{i_1+1 \leq i \leq 2n} dx_i \prod_{i_1+1 \leq i \leq 2n+1} C_j(\sigma, x_{i-1}, x_i), \\ C &= \int \prod_{2 \leq i \leq i_1-1} dx_i \prod_{2 \leq i \leq i_1} C_j(\sigma, x_{i-1}, x_i) \prod_{i=1}^{2n} d\bar{x}_i \prod_{1 \leq i \leq 2n+1} \bar{C}_j(\sigma, \bar{x}_{i-1}, \bar{x}_i). \end{aligned} \quad (33)$$

It leads, using again the norm bounds of type (22) on the “trapped loop” in the first part of C , to a bound

$$|\lambda|^{1/2} K \left(\Gamma_L(x, y) + M^{4j} \int dx_1 e^{-cM^j|x-x_1|} \Gamma_r(x_1, y) \right) \quad (34)$$

for some $r < L$. Applying the induction hypothesis concludes to the bound (24).

Finally for the second term we Wick contract again the σ field. There are again two subcases: the Wick contraction $\frac{\delta}{\delta\sigma}$ hits either a C_j or a \bar{C}_j . Let us call i the number of half-lines, either on the upper or on the lower circles, which are inside the Wick contraction, and x_{i_1}, \dots, x_{i_k} or $\bar{x}_{i_1}, \dots, \bar{x}_{i_k}$ the positions of the dotted lines *crossed* by the Wick contraction.

We have now two additional difficulties compared to the $L = 0$ case:

- we have to sum over where the Wick contraction hits, hence sum over i (because the Wick contraction creates a loop, hence potentially dangerous combinatoric). The solution is that the norm bound on the “trapped loop” in the C term of (27) erases more and more coupling constants as the loop gets longer: this easily pays for choosing the Wick contraction.
- the dotted lines *crossed* by the Wick contraction should be kept in the A term in inequality (27). In other words they become vertical lines at the next step, even if no vertical line was present in the initial graph. This is why we had to extend our induction to the category of GM graphs. This extension is what solves this difficulty.

We decompose the amplitude of the graph in the first subcase of Figure 7 as

$$\sum_i \int dz dx_{i_1}, \dots, dx_{i_k} C_j(x, z) T L_{x_{i_1}, \dots, x_{i_k}}(z, z) R_{x_{i_1}, \dots, x_{i_k}}(z, y) \bar{S}(x, y) \quad (35)$$

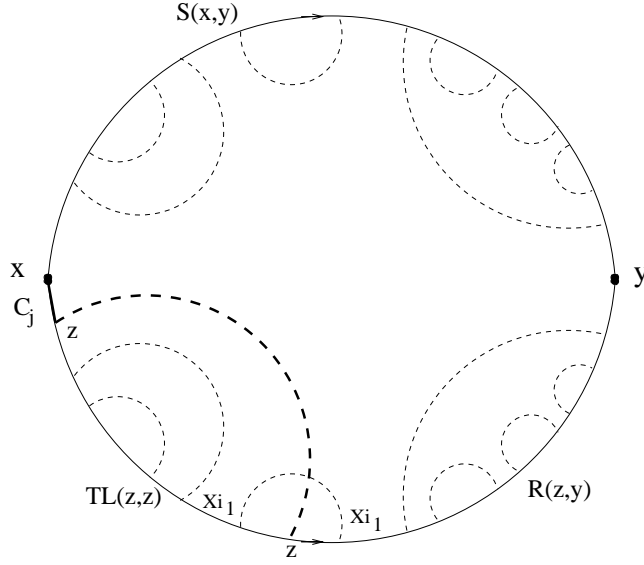


Figure 7: The Wick contraction

with hopefully straightforward notations, and we apply the Schwarz inequality (27), with:

$$\begin{aligned}
A &= |\lambda|^{i/8} \sum_i \int dz dx_{i_1}, \dots, dx_{i_k} \int C_j(x, z), \\
B &= R_{x_{i_1}, \dots, x_{i_k}}(z, y), \\
C &= |\lambda|^{-i/8} TL_{x_{i_1}, \dots, x_{i_k}}(z, z) \bar{S}(x, y).
\end{aligned} \tag{36}$$

Now the first remark is that $i|\lambda|^{i/8}$ is bounded by K for small λ so we need only to find a uniform bound at fixed i .

The $A|B|^2$ is a convolution of an explicit propagator bounded by (1) with a new GM graph (with vertical lines which are the crossed lines at x_{i_1}, \dots, x_{i_k}) either identical to G or shorter. If it is shorter we apply the induction hypothesis. If it is not shorter we obtain a convolution equation term like in the right hand side of (24).

The $A|C|^2$ contains a trapped loop TL with i vertices. Each half-vertex of the trapped loop has only $|\lambda|^{1/8}$ because of the $|\lambda|^{-i/8}$ factor in (36). The trapped loop is again of the GM nature with vertical lines which are the crossed lines at x_{i_1}, \dots, x_{i_k} . But we can still apply the bound (22) to this trapped loop. Therefore the bound on the sum of the $A|B|^2$ and $A|C|^2$ is again of the type (34).

Finally the second subcase, where the Wick contraction $\frac{\delta}{\delta\sigma}$ hits a \bar{C}_j , is exactly similar, except that the “almost trapped loop” is now something of the type $\bar{TL}(x, z)$ rather than $TL(z, z)$. But the bound (22) also covers this case,

so that everything goes through.

Collecting the bounds (34) in every case completes the proof of Lemmas 4.3 and 4.4 for Γ_{L+1} . This concludes the proof of Lemmas 4.3 and 4.4 for all L .

5 Further topics

5.1 Higher functions

The analysis of the $2p$ point functions is similar to that of the previous section. The general $2p$ point function $S^c(x_1, \dots, x_{2p})$ defined by (14) contains p resolvents of the $C_j(\sigma)$ type and a certain number of loop vertices joining or decorating them. Turning around the tree we can still identify the drawing as a set of decorated resolvents joined by local vertices or dotted lines as in Figures 8 and 9, which are the analogs of Figures 3 and 4. This is because any chain of loop vertices joining resolvents can be “absorbed” into decorations of one of these resolvents.

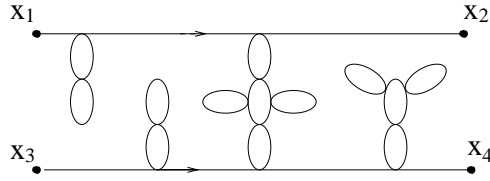


Figure 8: A connected 4 point function

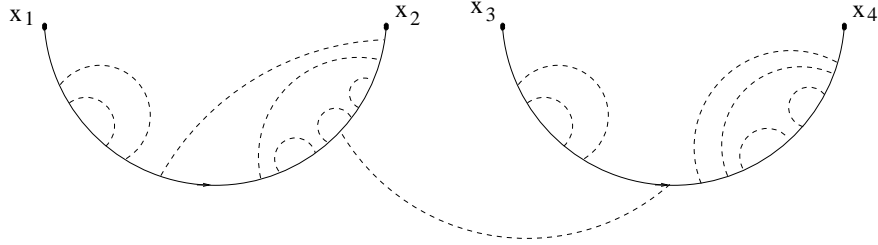


Figure 9: The “half-disk” representation of that connected 4 point function

The factor $2p!$ in (15) can be understood as a first factor $2p!!$ to choose the pairing of the points in p resolvents and an other $p!$ for the choice of the tree of connecting loop vertices between them. We can again bound each term of the initial expansion by a “mirror” term pointwise positive in σ with p disks as shown in Figure 10.

A Lemma similar to Lemma 4.1 is again proved by a bound on generalized mirror graphs such as those of Figure 10 but with additional vertical lines inside the p disks. This bound is proved inductively by a single resolvent step followed

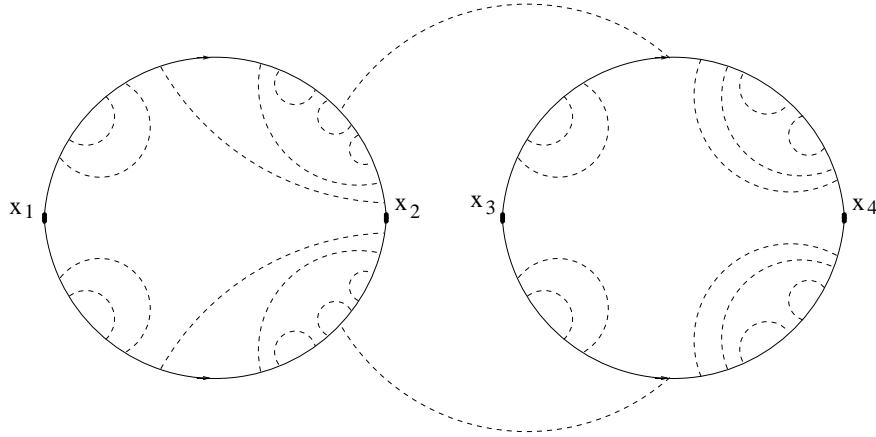


Figure 10: The mirror representation of the same connected 4 point function

by a Fredholm bound similar to Lemmas 4.3 and 4.4. Verifications are left to the reader.

5.2 Large/small Field Expansion

To prove the tree decay of the $2p$ -point connected functions as external arguments are pulled apart, it is possible to replace the Fredholm inequality of the previous section by a so-called *large/small field expansion*. It still relies on a resolvent expansion, but integration by parts is replaced by a probabilistic analysis over σ . We recall only the main idea, as this expansion is explained in detail in [15, 20] but also in a very large number of other earlier publications.

A lattice \mathcal{D} of cubes of side M^{-j} is introduced and the expansion is

$$1 = \prod_{\Delta \in \mathcal{D}} \left\{ \chi \left(\int_{\Delta} M^{4j} |\lambda|^{\epsilon} \sigma^2(x) dx \right) + [1 - \chi \left(\int_{\Delta} M^{4j} |\lambda|^{\epsilon} \sigma^2(x) dx \right)] \right\} \quad (37)$$

where χ is a function with compact support independent of j and λ .

The small field region S is the union of all the cubes for which the χ factor has been chosen. The complement, called the large field region L , is decomposed as the union of connected pieces L_k . Each such connected large field region has a small probabilistic factor for each of its cube using e.g. some standard Tchebycheff inequality.

The field is decomposed according to its localization as $\sigma = \sigma_S + \sum_k \sigma_{L_k}$. Then the resolvent $C_j(\sigma, x, y)$ is simply bounded in norm if x and y belong to the same L_k region because the decay is provided by the probabilistic factor associated to L_k .

The σ_S piece is expanded according to resolvent formulas such as

$$C_j(\sigma_S, x, y) = C_j(x, y) - i\sqrt{\lambda} \int dz C_j(x, z) \sigma_S(z) C_j(\sigma_S, z, y), \quad (38)$$

which can be iterated to infinity because the σ_S field is not integrated with the Gaussian measure but bounded with the help of the small field conditions.

Then inside each connected large field region L_k the resolvent $C_j(\sigma_{L_k}, x, y)$ is simply bounded in norm. The decay is provided by the probabilistic factor associated to L_k . Between different connected large field regions, the decay is provided by the small field resolvent expansion.

However one advantage of the loop expansion presented in this paper is to avoid the need of any lattice of cubes for cluster/Mayer expansions. If possible, it seems better to us to avoid reintroducing a lattice of cubes in such a small/large field analysis.

5.3 Multiscale Expansions

The result presented in this paper for a single scale model should be extended to a multiscale analysis. This means that every loop-vertex or resolvent should carry a scale index j which represents the *lowest* scale which appears in that loop or resolvent. Then we know that the forest formula used in this paper should be replaced by a so-called “jungle” formula [6] in which links are built preferentially between loop vertices and resolvents of highest possible index.

This jungle formula has to be completed by a “vertical expansion” which tests whether connected contributions of higher scales have less or more than four external legs of lower scales, see e.g. [15]. A renormalization expansion then extracts the local parts of the corresponding two and four point contributions and resums them into effective couplings. In this way it should be possible to finally complete the program [15] of a Bosonic renormalization-group-resummed expansion whose pieces are defined through totally explicit formulas without using any induction. Indeed the missing ingredient in [15], namely an explicit formula to insert *Mayer expansions* between each cluster expansion, would be totally avoided. The new multiscale expansion would indeed not require any cluster nor Mayer expansion at any stage.

The expansion would be completed by auxiliary resolvent expansions, either with integration by parts in the manner of section 4 or with a small/large field analysis as in subsection 5.2 above. This is necessary to establish scaled spatial decay, which in turn is crucial to prove that the renormalized two and four point contributions are small. But these new auxiliary expansions shall be used only to prove the desired bounds, not to define the expansion itself.

5.4 Vector Models

The method presented here is especially suited to the treatment of large N vector models. Indeed we can decompose a vector ϕ^4 interaction with an intermediate scalar field as in (2) but in such a way that the flow of vector indices occurs within the loop-vertices. Every loop vertex simply carries therefore a global N factor where N is the number of colors. Hence we expect that the loop expansion presented here is the right tool to glue different regimes of the renormalization group governed respectively e.g. in the ultraviolet regime by

a small coupling expansion and in the infrared regime by a “non-perturbative” large N expansion of the vector type. This gluing problem occurs in many different physical contexts, from mass generation of the two-dimensional Gross-Neveu [20] or non-linear σ -model [21] to the BCS theory of superconductivity [22]. These gluing problems have been considered until now too complicated in practice for a rigorous constructive analysis.

5.5 Matrix models and $\phi_4^{\star 4}$

The loop expansion is also suited for the treatment of large N matrix models and was in fact found for this reason [17]. Our first goal is to apply it to the full construction of non-commutative $\phi_4^{\star 4}$ [18], either in the so-called matrix base [23, 24] or in direct space [25].

One needs again to develop for that purpose the multiscale version of the expansion and the resolvent bounds analogs to section 4 or subsection 5.2 above. Indeed neither the matrix propagator nor the Mehler x space propagator are diagonal in the corresponding representations/footnoteThere is an interesting exception: the matrix propagator of $\phi_4^{\star 4}$ becomes diagonal in the matrix base at the very special ultraviolet fixed point where Ω , the Grosse-Wulkenhaar parameter, is 1, Of course the general non-diagonal case has to be treated..

Ultimately we hope that better understanding the non-commutative models of the matrix or quasi-matrix type should be useful in many areas of physics, from physics beyond the standard model [26, 27, 28] to more down to earth physics such as quark confinement [29] or the quantum Hall effect [30].

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